Entanglement and topological entropy of the toric code at finite temperature

Claudio Castelnovo[1] and Claudio Chamon[2]
[1] Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Oxford, OX1 3NP, UK
[2] Physics Department, Boston University, Boston, MA 02215, USA
(Dated: February 1, 2008)

We calculate exactly the von Neumann and topological entropies of the toric code as a function of system size and temperature. We do so for systems with infinite energy scale separation between magnetic and electric excitations, so that the magnetic closed loop structure is fully preserved while the electric loop structure is tampered with by thermally excited electric charges. We find that the entanglement entropy is a singular function of temperature and system size, and that the limit of zero temperature and the limit of infinite system size do not commute. The two orders of limit differ by a term that does not depend on the size of the boundary between the partitions of the system, but instead depends on the topology of the bipartition. From the entanglement entropy we obtain the topological entropy, which is shown to drop to half its zero-temperature value for any infinitesimal temperature in the thermodynamic limit, and remains constant as the temperature is further increased. Such discontinuous behavior is replaced by a smooth decreasing function in finite-size systems. If the separation of energy scales in the system is large but finite, we argue that our results hold at small enough temperature and finite system size, and a second drop in the topological entropy should occur as the temperature is raised so as to disrupt the magnetic loop structure by allowing the appearance of free magnetic charges. We discuss the scaling of these entropies as a function of system size, and how the quantum topological entropy is shaved off in this two-step process as a function of temperature and system size. We interpret our results as an indication that the underlying magnetic and electric closed loop structures contribute equally to the topological entropy (and therefore to the topological order) in the system. Since each loop structure *per se* is a classical object, we interpret the quantum topological order in our system as arising from the ability of the two structures to be superimposed and appear simultaneously.

I. INTRODUCTION

Some strongly correlated quantum systems have rather rich spectral properties, such as ground state degeneracies that are not related to symmetries, but instead to topology.[1,2] Such systems are said to be topologically ordered,[3] and they can have excitations with fractionalized quantum numbers,[4] as in the case of the fractional quantum Hall states. There have been proposals to utilize topologically ordered states for fault tolerant quantum computation, exploiting the resilience of these systems to decoherence by local perturbations or disturbances by the environment.

Levin and Wen[5], and Kitaev and Preskill[6] recently proposed that a characteristic signature of topological order can be found in a subleading correction of the Von Neumann (entanglement) entropy in systems prepared in (one of) its ground state(s). This topological correction to the entanglement entropy was indeed confirmed by exact calculations in discrete models exhibiting topological order, as well as in continuum systems such as fermionic Laughlin states.[7] The notion of topological entropy provides a "non-local order parameter" for topologically ordered systems. Hereafter, we refer to topological order as characteristically identified by such non-vanishing topological entropy.

Although quantum topological order was introduced as a pure zero-temperature concept, is was recently shown[8] that a closely related behavior can be observed also in mixed state density matrices that describe classical systems in the presence of hard constraints. These findings show that topological order can survive thermal mixing under certain conditions, e.g., in hard constrained systems. Moreover, any possible experimental observation of quantum topological order must take into account the fact that the $T = 0$ limit is only an idealization and temperature, albeit small, is a perturbation that cannot be neglected. This is particularly relevant, for example, if one is interested in a practical application of topological order towards quantum computing, which will always be done at finite temperature. It is therefore interesting to study the behavior of topologically ordered systems as the temperature is gradually raised from zero, in search of a unified picture of topological order encompassing both the quantum zero-temperature limit and the classical hard-constrained limit.

In this paper, we investigate the fate of quantum topological order in the two-dimensional toric code on the square lattice in thermal equilibrium with a bath at finite temperature. In particular, we do so by studying the entanglement and topological entropies of the system, which we compute exactly.

We start from the zero-temperature limit of the model, which has been thoroughly studied in Ref. 9. In this limit, the ground state (GS) of the system can be mapped onto two loop structures[10] each of which, we argue, is responsible for half of the topological contribution to the von Neumann entropy (i.e., half of the topological entropy of the system). As temperature is raised from zero, thermal equilibration disrupts (breaks) the loop structure and it is expected to destroy topological order. With an exact calculation in the limit where one of the two loop structures is fully preserved while the other is allowed to thermalize,[11] we show that the topological entropy gradually decreases as a function of temperature, for fixed and finite system size, from its zero-temperature value down to precisely half of that value. In particular, the temperature dependence of the topological entropy can be shown to appear always through the product $K_A(T)\,N$, where $K_A(T)$ is a monotonic function of temperature with $K_A(0) = 0$ and $K_A(\infty) = \infty$,

and N is an extensive quantity that scales linearly with the number of degrees of freedom in the system. Therefore, the thermodynamic limit $N \to \infty$ and the $T \to 0$ limit *do not commute*, and if the former is taken first, the topological entropy becomes a singular function at $T = 0$, and it equals one half of its zero-temperature value for any $T \neq 0$. In other words, in the thermodynamic limit any infinitesimal temperature is able to fully disrupt any loop structure for which we allow thermalization, and the contribution from this structure to the topological entropy is completely lost (irrespective of the presence of a finite energy gap). On the other hand, finite size systems can retain a statistical contribution to the topological entropy (in the sense that its value varies continuously with temperature) originating from a thermalized underlying loop structure.

From our results, we then infer the behavior of the finite-temperature topological entropy in the generic case, as illustrated in Fig. 1. For finite size systems, we expect to ob-

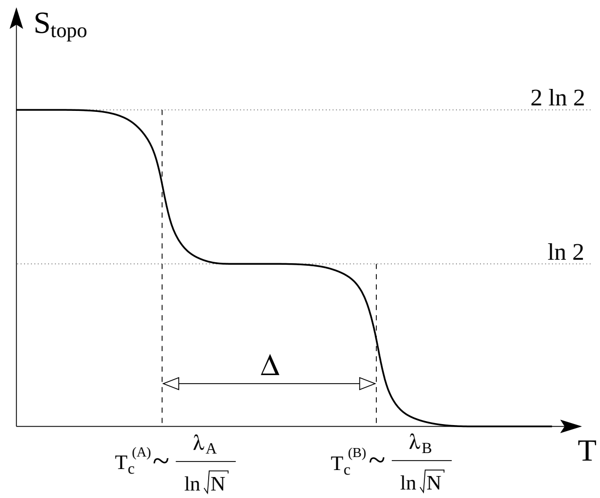

FIG. 1: Qualitative behavior of the topological entropy as a function of temperature T and number of degrees of freedom $2N$, for the generic case where the two coupling constants in the model are well separated, i.e., $\lambda_A \ll \lambda_B$. The exact shape of the first crossover is shown in Fig. 5

serve two continuous decays of the topological entropy, due to the gradual disruption of each of its two loop contributions. Each drop occurs when the number of corresponding defects $\sim N\ e^{-2\lambda_{A,B}/T}$ reaches a value of order one, where $2N$ is the number of degrees of freedom in the system, and λ_A and λ_B are the two coupling constants in the model, associated with one loop structure each. The separation between the two decays (the quantity Δ in the figure) is therefore proportional to the difference between the two coupling constants λ_A and λ_B in the Hamiltonian. Once again, if the thermodynamic limit is taken first, both decays collapse into a singular behavior where the topological entropy vanishes everywhere except for $T = 0$, where its value depends on the order between the thermodynamic and zero-temperature limit. Notice also that, although the crossover temperature $T_{\text{cross}}^{(A,B)} \sim \lambda_{A,B} / \ln \sqrt{N}$ goes to zero in the limit of $N \to \infty$, it does so only in a logarithmic fashion.

The case of classical topological order is recovered in the present study when thermal fluctuations are allowed to completely break one of the loop structures while the other is strictly preserved. Indeed, this can be accomplished by imposing appropriate (local) hard constraints on the classical analog of the toric code.[8] Our results illustrate how both the concept of quantum topological order and of classical topological order are equally fragile, as they truly exist only in the zero-temperature / hard-constraint limit. Their effects however can extend well into the finite-temperature / soft-constraint realm – as our calculations show – so long as the size of the system is finite.

Our results suggest a simple pictorial interpretation of quantum topological order, at least for systems where there is an easy identification of loop structures as in the case here studied. The picture is that (i) the two loop structures contribute equally and independently to the topological order at zero temperature; (ii) each loop structure *per se* is a classical (non-local) object carrying a contribution of $\ln D$ to the topological entropy ($D = 2$ being the so-called quantum dimension of the system); and (iii) the quantum nature of the zero-temperature system resides in the fact that two independent loop structures can be superimposed (therefore leading to an overall topological entropy equal to $2 \ln D = \ln D^2$). In this sense, our results lead to an interpretation of quantum topological order, at least for systems with simple loop structures, as the quantum mechanical version of a classical topological order (given by each individual loop structure).

We also investigate the von Neumann (entanglement) entropy S_{VN} as a function of temperature and system size. For instance, we show that, given any bipartition $(\mathcal{A}, \mathcal{B})$ of the whole system $\mathcal{S} = \mathcal{A} \cup \mathcal{B}$, the quantity

$$\begin{aligned} \Delta S &= \lim_{T\to 0, L\to\infty} S_{\text{VN}}^{\mathcal{A}}(T) - \lim_{L\to\infty, T\to 0} S_{\text{VN}}^{\mathcal{A}}(T) \\ &= (m_{\mathcal{B}} - 1) \ln 2, \end{aligned} \quad (1)$$

where $S_{\text{VN}}^{\mathcal{A}}(T)$ is the entropy of partition $\mathcal{A}$ after tracing out partition $\mathcal{B}$, $m_{\mathcal{B}}$ is the number of disconnected regions in $\mathcal{B}$, and $L = \sqrt{N}$ is the linear size of the system. From this result, we learn that the topological contribution to the entanglement entropy can be filtered out directly from a single bipartition, provided $m_{\mathcal{B}} > 1$, as opposed to the constructions in Refs. 5,6 that require a linear combination over multiple bipartitions.

We also show that, as soon as the temperature is different from zero, the von Neumann entropy is no longer symmetric upon exchange of subsystem $\mathcal{A}$ and subsystem $\mathcal{B}$, and it acquires a term that is extensive in the number of degrees of freedom that have not been traced out (see Eq. (39)). Symmetry and dependence only on the boundary degrees of freedom, at least in the thermodynamic limit, can be recovered if one considers instead the mutual information

$$I_{\mathcal{A}\mathcal{B}}(T) = \frac{1}{2} \left[S_{\text{VN}}^{\mathcal{A}}(T) + S_{\text{VN}}^{\mathcal{B}}(T) - S_{\text{VN}}^{\mathcal{A}\cup\mathcal{B}}(T) \right], \quad (2)$$

as we explicitly show in this paper. Notice that $I_{\mathcal{A}\mathcal{B}}(0) \equiv S_{\text{VN}}^{\mathcal{A}}(0) = S_{\text{VN}}^{\mathcal{B}}(0)$. Once again, the mutual information exhibits a singular behavior at zero temperature since

the thermodynamic limit and the zero-temperature limit do not commute. We find that the explicit topological contribution to $I_{\mathcal{AB}}(T)$ in the thermodynamic limit is $-\frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\ln 2$, where $m_{\mathcal{A}}$ ($m_{\mathcal{B}}$) is the number of disconnected components of partition $\mathcal{A}$ ($\mathcal{B}$).

Although we consider here a very specific model, we believe that our results are of relevance to a broader context, at least at a qualitative level. For example, it would be interesting to investigate the specific behavior of systems where the underlying structures responsible for the presence of topological order are no longer identical to each other. This is the case of the three-dimensional extension of the toric code, where a closed loop structure becomes dual to a closed membrane structure.[12]

The paper is organized as follows. In Section II we present the model and discuss its characteristic features and properties. In Section III we compute the von Neumann entropy of the system as a function of temperature and system size, in the limit of one of the coupling constants going to infinity. We then obtain the exact expression for the topological entropy in Sec. IV, and we illustrate its behavior with numerical results. Finally, we discuss the implications of our results for the system with finite coupling constants and we infer the full temperature and system size dependence of the topological entropy in Sec. V. Conclusions are drawn in Sec. VI.

II. THE FINITE-TEMPERATURE TORIC CODE

The zero-temperature limit of the model considered here was studied by Kitaev in Ref. 9. It can be represented by $2N$ spin-$1/2$ degrees of freedom on the bonds of an $L \times L$ square lattice, $N = L^2$, with periodic boundary conditions (toric geometry). The system is endowed with a Hamiltonian that can be written in terms of star and plaquette operators as

$$H = -\lambda_B \sum_{\text{plaquettes } p} B_p - \lambda_A \sum_{\text{stars } s} A_s, \tag{3}$$

where λ_A and λ_B are two positive coupling constants, $B_p = \prod_{i\in p} \sigma_i^{\rm z}$, and $A_s = \prod_{j \in s} \sigma_j^{\rm x}$, with i labeling all four edges of plaquette p and j labeling all four bonds meeting at vertex s of the square lattice. Notice that the Hamiltonian, all the B_p operators and all the A_s commute with each other, and one can diagonalize them simultaneously. Given that there are $N-1$ independent plaquette operators and $N-1$ independent star operators ($\prod_{p=1}^{N} B_p = \mathbf{1} = \prod_{s=1}^{N} A_s$), the eigenvectors with fixed B_p and A_s quantum numbers form a 2^2-dimensional space. Furthermore, one can show that the GS 4-fold degeneracy has a topological nature that can be split only by the action of non-local (system spanning) operators. The ground state wavefunctions of this model are known exactly,[9] and can be written in the $\sigma^{\rm z}$ basis as

$$|\Psi_0\rangle = \frac{1}{|G|^{1/2}} \sum_{g\in G} g|0\rangle, \tag{4}$$

where G is the Abelian group generated by all star operators $\{A_s\}_{s=1}^{N}$, modulo the fact that $\prod_{s=1}^{N} A_s = \mathbf{1}$, $|G| = 2^{N-1}$ is the dimension of G, and $|0\rangle = |\sigma_1^{\rm z} \,\ldots\, \sigma_n^{\rm z}\rangle$ is any given state that satisfies the condition $B_p|0\rangle = |0\rangle, \forall\, p$. There are four inequivalent choices for $|0\rangle$, corresponding to the four different topological sectors of the model. The choice of sector is immaterial to the results presented hereafter, since they all have the same entanglement,[13] and we will set $|0\rangle = |++\ldots+\rangle$ for convenience throughout the rest of the paper.

Notice that the system is symmetric upon exchange of $\sigma^{\rm x}$ with $\sigma^{\rm z}$ components and of stars with plaquettes on the lattice. In the $\sigma^{\rm x}$-basis, each site must have 0, 2 or 4 spins with a negative $\sigma^{\rm x}$ component on the adjacent bonds. If we were to remove all the bonds with a negative $\sigma^{\rm x}$ component, we would obtain a configuration of closed loops on the square lattice, where loops are allowed to cross but do not overlap. Once a convention is established on how to interpret sites entirely surrounded by spins with positive $\sigma^{\rm x}$ component (e.g., as two different loop parts touching at the corner, say the up-right and down-left loops), then one can establish a one-to-one correspondence between all basis states and all loop configurations on the square lattice where loops cannot overlap and can at most touch at a corner in an up-right, down-left fashion (see Fig. 2). The same is true for the $\sigma^{\rm z}$-basis, but the loops

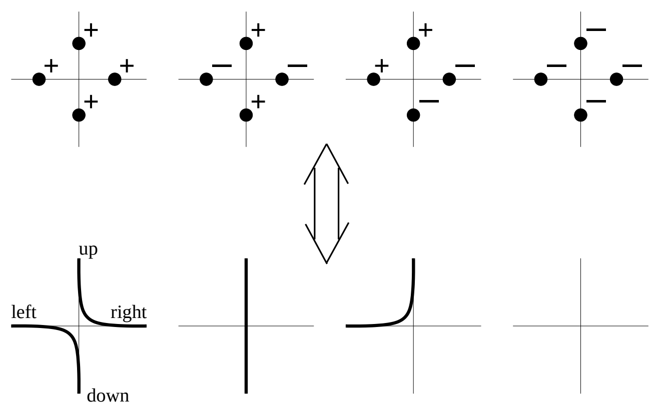

FIG. 2: Illustration of the spin-loop correspondence discussed in the text. All vertices with two positive and two negative $\sigma^{\rm x}$ components on the adjacent links can be obtained via appropriate rotations of the ones shown in the figure.

now live on the dual lattice given by the centers of the plaquettes of the original lattice. This description of the GS of the toric code in terms of loop degrees of freedom gives a qualitative picture of the origin of the non-local behavior of the system from which the presence of topological order stems. In particular, given that the two simultaneous loop descriptions can be mapped one onto the other upon exchanging $\sigma^{\rm x}$ with $\sigma^{\rm z}$ as well as the square lattice with its dual, it is tempting to speculate that they contribute equally to the topological order present in the system, and each loop structure is responsible for precisely half of the topological entropy. Our exact calculations show that this naive picture is indeed correct: if either of the loop structures is exactly preserved while the other is destroyed, e.g., via coupling to a thermal bath, the topological entropy of the system lowers to half of its original value.

Based on the $\sigma^{\rm x}$ and $\sigma^{\rm z}$ loop description, all possible per-

turbations to the system can be qualitatively divided into three different classes: (i) those that couple to a σ^z-like term, and are able – if sufficiently strong – to disrupt the underlying σ^x loop structure, but not the σ^z one; (ii) those that couple to a σ^x-like term, with precisely the opposite effect; and (iii) those that couple to a σ^y-like term, and are able – again, if the coupling constant is large enough – to disrupt both loop structures, thus leading to a vanishing topological entropy. A generic coupling to a thermal bath is likely to encompass all of the above terms and in the thermodynamic limit the vanishing of the topological entropy is unavoidable. For finite size systems and at low enough temperature, however, the relative scale of the two coupling constants λ_B and λ_A plays a crucial role in determining how effective each of the above terms is with respect to the others. In this paper we consider the case when the two energy scales are well separated, namely $\lambda_B \gg \lambda_A$, and we discuss qualitatively the behavior of the system as the separation becomes weaker and vanishes. A large separation between the two energy scales is indeed expected if we notice that the toric code is a lattice realization of a $\mathbb{Z}_2$ gauge theory, where the two coupling constants λ_A and λ_B relate directly to the chemical potential of the electric and magnetic monopoles.[14] On the ground of a large separation between the two energy scales in the Hamiltonian, three distinct temperature regimes can be outlined:

(a) $T \ll \lambda_A / \ln\sqrt{N}$, when all thermal excitations have a small Boltzmann weight and for finite size systems at finite time scales the topological entropy effectively retains its zero temperature value because of the scarcity of defects that can disrupt the loop structure;

(b) $\lambda_A / \ln\sqrt{N} < T \ll \lambda_B / \ln\sqrt{N}$, when thermal excitations of the σ^z type can disrupt the σ^x loops structure, while the σ^x-like excitations are rather unlikely to occur and they can be effectively neglected; and

(c) $\lambda_B / \ln\sqrt{N} < T$, when the appearance of all the three types of thermal excitations leads to the complete disruption of the topological contribution to the entanglement entropy.

(Notice that the opposite case, where $\lambda_A \gg \lambda_B$, leads to equivalent results based on the symmetry of the model.) The temperature range considered in this paper corresponds to regimes (a) and (b), where the σ^z loop structure is effectively preserved for sufficiently small system sizes and time scales. We will then discuss how our results can be used to infer the behavior of the topological entropy across the whole temperature range, illustrated in Fig. 1.

Basically, one can define a temperature dependent defect separation length scales $\xi_{A,B} \sim e^{\lambda_{A,B}/T}$, so that as long as the sample size is below these corresponding scales, the system is free from the associated type of defects. Similarly, one can define temperature-dependent time scales for defects to appear. The toric code is *fragile* in the sense that $\mathcal{O}(1)$ defects destroy its topological order, so that for practical considerations not only the temperature must be small compared to a gap, but the system size and the time scales must not be too large as well.

We now consider the simplification where the finite system length $\sqrt{N} \ll \xi_B$, so we can neglect defects in the σ^z loop structure. Forbidding any defects in the σ^z loop structure is equivalent to neglecting all thermal processes that violate the constraint $\prod_{i\in p} \sigma_i^z = +1, \forall\, p$. Therefore, the Hilbert space in the regime of interest and within the chosen topological sector (recall that $|0\rangle = |+ + \ldots +\rangle$) is given by $\{g|0\rangle \mid g \in G\}$. The equilibrium properties of the system are then captured by the finite-temperature density matrix

$$\begin{aligned} \rho(T) &= \frac{1}{Z}\, e^{-\beta \hat{H}} \\ &= \frac{\sum_{g,g'\in G} \langle 0|g e^{-\beta H} g g'|0\rangle \; g|0\rangle\langle 0|g g'}{\sum_{g\in G} \langle 0|g e^{-\beta H} g|0\rangle}, \end{aligned} \tag{5}$$

where we used the group property to write a generic element $g'' \in G$ as $g'' = g g'$, $\exists!\, g' \in G$ given $g \in G$. Recall that all group elements, as well as their composition, are defined as products of star operators modulo the identity $\prod_{s=1}^N A_s = \mathbf{1}$.

For the model under consideration, it is convenient to rewrite the Hamiltonian (3) as

$$\begin{aligned} H &= -\lambda_B P - \lambda_A S \\ P &= \sum_{\text{plaquettes } p} B_p \\ S &= \sum_{\text{stars } s} A_s . \end{aligned} \tag{6}$$

Notice that $P g|0\rangle = N g|0\rangle, \forall\, g \in G$, and therefore

$$\begin{aligned} \langle 0|g e^{-\beta H} g g'|0\rangle &= e^{\beta\lambda_B N} \langle 0|g e^{\beta\lambda_A S} g g'|0\rangle \\ &= e^{\beta\lambda_B N} \langle 0|e^{\beta\lambda_A S} g'|0\rangle \end{aligned} \tag{7}$$

where we used the fact that any g commutes with S by construction.

Now, recall the definition of a group element $g \in G$, which can be symbolically represented by the notation $g \equiv \prod_{s\in g} A_s$ (modulo the identity $\prod_{s=1}^N A_s = \mathbf{1}$, i.e., $g = \prod_{s\in g} A_s = \prod_{s\notin g} A_s$). Given the expansion

$$e^{\beta\lambda_A S} = \prod_{\text{stars } s} \left[\cosh\beta\lambda_A + \sinh\beta\lambda_A\, A_s\right], \tag{8}$$

which follows from the definition $S = \sum_{s=1}^N A_s$ and from the fact that $A_s^2 \equiv \mathbf{1}$, one obtains

$$\begin{aligned} &\langle 0|e^{\beta\lambda_A S} g'|0\rangle = \\ &\quad = \langle 0| \prod_s \left[\cosh\beta\lambda_A + \sinh\beta\lambda_A\, A_s\right] \prod_{s'\in g'} A_{s'} |0\rangle \\ &\quad = \left(\cosh\beta\lambda_A\right)^N \left(\tanh\beta\lambda_A\right)^{n(g')} \\ &\qquad + \left(\cosh\beta\lambda_A\right)^N \left(\tanh\beta\lambda_A\right)^{N-n(g')}, \end{aligned} \tag{9}$$

where N is the total number of stars in the system, and $n(g')$ is the number of flipped stars in g'. Notice that the ambiguity in the definition of $g = \prod_{s\in g} A_s \equiv \prod_{s\notin g} A_s$ – namely the

fact that if g' is given by the product of a set of A_s, it is also given by the product of all other A_s but for those in the set – does not affect the equation above. In fact, this ambiguity amounts to the mapping $n(g') \leftrightarrow N - n(g')$. Similarly,

$$\begin{aligned} Z &= \sum_{g\in G} \langle 0| g e^{-\beta H} g |0\rangle \\ &= |G|\, e^{\beta\lambda_B N} \langle 0| \prod_{\text{stars } s} \Big[\cosh\beta\lambda_A + \sinh\beta\lambda_A\, A_s\Big] |0\rangle \\ &= |G|\, e^{\beta\lambda_B N} (\cosh\beta\lambda_A)^N \Big[1 + (\tanh\beta\lambda_A)^N\Big]. \end{aligned} \tag{10}$$

Substituting Eq. (9) and Eq. (10) into Eq. (5) after relabeling $K_A = -\ln[\tanh(\beta\lambda_A)]$ gives

$$\rho(T) = \sum_{g,g'\in G} \frac{1}{|G|} \frac{\Big[e^{-K_A n(g')} + e^{-K_A\big(N - n(g')\big)}\Big]}{\big[1 + e^{-K_A N}\big]}\, g|0\rangle\langle 0| g g'. \tag{11}$$

In the limit of $T \to 0$ ($\beta \to \infty$), $K_A \to 0^+$, all g' are equally weighed, and one recovers the density matrix of the zero-temperature Kitaev model. In the limit $T \to \infty$ ($\beta \to 0$), $K_A \to \infty$, all g' are exponentially suppressed except for $g' = \mathbf{1}$, and one recovers the mixed-state density matrix of the topologically ordered classical system discussed in Ref. 8.

III. THE VON NEUMANN ENTROPY

Let us consider a generic bipartition of the system $\mathcal{S}$ into subsystem $\mathcal{A}$ and subsystem $\mathcal{B}$ ($\mathcal{S} = \mathcal{A} \cup \mathcal{B}$). Let us also define $\Sigma_{\mathcal{A}}$ ($\Sigma_{\mathcal{B}}$) to be the number of star operators A_s that act solely on spins in $\mathcal{A}$ ($\mathcal{B}$), and $\Sigma_{\mathcal{AB}}$ as the number of star operators acting simultaneously on both subsystems. Clearly these quantities satisfy the relationship $\Sigma_{\mathcal{A}} + \Sigma_{\mathcal{B}} + \Sigma_{\mathcal{AB}} = N$. Whenever a partition is made up of multiple connected components, e.g., $\mathcal{A} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{m_{\mathcal{A}}}$ with $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ and $\mathcal{A}_i$ connected, $\forall\, i, j$, let us denote with $\Sigma_{\mathcal{A}_i}$ the number of star operators acting solely on $\mathcal{A}_i$ ($\Sigma_{\mathcal{A}} = \sum_i \Sigma_{\mathcal{A}_i}$). [Since in the following we will consider only the case where either $\mathcal{A}$ or $\mathcal{B}$ have multiple connected components, but not both at the same time, $\Sigma_{\mathcal{AB}_i}$ will be used unambiguously to denote the number of star operators acting simultaneously on $\mathcal{A}_i$ and on $\mathcal{B}$ or on $\mathcal{B}_i$ and on $\mathcal{A}$, according to the specific case ($\Sigma_{\mathcal{AB}} = \sum_i \Sigma_{\mathcal{AB}_i}$).]

The von Neumann (entanglement) entropy S_{VN} of a bipartition $(\mathcal{A}, \mathcal{B})$ is given by

$$S_{\text{VN}}^{A} \equiv -\text{Tr}\left[\rho_A \ln \rho_A\right] = S_{\text{VN}}^{B}, \tag{12}$$

where $\rho_A = \text{Tr}_{\mathcal{B}}(\rho)$ is the reduced density matrix obtained from the full density matrix ρ by tracing out the degrees of freedom of subsystem B, and the last equality holds whenever the full density matrix ρ is a pure-state density matrix.

In order to compute the von Neumann entropy (12) from the finite-temperature density matrix (11), we first obtain the reduced density matrix of the system using the same approach of Ref. 13,

$$\begin{aligned} \rho_{\mathcal{A}}(T) &= \sum_{g,g'\in G} \frac{1}{|G|} \frac{\Big[e^{-K_A n(g')} + e^{-K_A\big(N - n(g')\big)}\Big]}{\big[1 + e^{-K_A N}\big]} \times \\ &\qquad g_{\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}| g_{\mathcal{A}} g'_{\mathcal{A}}\, \langle 0_{\mathcal{B}}| g_{\mathcal{B}} g'_{\mathcal{B}} g_{\mathcal{B}} |0_{\mathcal{B}}\rangle \\ &= \sum_{g\in G, g'\in G_{\mathcal{A}}} \frac{1}{|G|} \frac{\Big[e^{-K_A n(g')} + e^{-K_A\big(N - n(g')\big)}\Big]}{\big[1 + e^{-K_A N}\big]} \times \\ &\qquad g_{\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}| g_{\mathcal{A}} g'_{\mathcal{A}} \\ &\equiv \frac{1}{|G|} \sum_{g\in G, g'\in G_{\mathcal{A}}} \eta_T(g')\, g_{\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}| g_{\mathcal{A}} g'_{\mathcal{A}}, \end{aligned} \tag{13}$$

where we used the generic tensor decomposition $|0\rangle = |0_{\mathcal{A}}\rangle \otimes |0_{\mathcal{B}}\rangle$, $g = g_{\mathcal{A}} \otimes g_{\mathcal{B}}$, and the fact that $\langle 0_{\mathcal{B}}| g_{\mathcal{B}} g'_{\mathcal{B}} g_{\mathcal{B}} |0_{\mathcal{B}}\rangle = 1$ if $g'_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}}$ and zero otherwise. The latter follows immediately from the fact that the group G is Abelian and that $A_s^2 = \mathbf{1}, \forall\, s$, and therefore $g_{\mathcal{B}}^2 = \mathbf{1}_{\mathcal{B}}$ for any choice of $\mathcal{B}$. We also denoted by $G_{\mathcal{A}} = \{g \in G \mid g_{\mathcal{B}} = \mathbf{1}_{\mathcal{B}}\}$ the subgroup of G given by all operations g that act trivially on $\mathcal{B}$ (similarly for $G_{\mathcal{B}}$ in the following). For convenience of notation we defined

$$\eta_T(g') = \frac{\Big[e^{-K_A n(g')} + e^{-K_A\big(N - n(g')\big)}\Big]}{\big[1 + e^{-K_A N}\big]}. \tag{14}$$

Notice that a star operator A_s can either act solely on spins in partition $\mathcal{A}$ (represented in the following by the notation $s \in \mathcal{A}$), solely on spins in partition $\mathcal{B}$ ($s \in \mathcal{B}$), or simultaneously on spins belonging to $\mathcal{A}$ and $\mathcal{B}$ (which we will refer to as *boundary star operators*, and represent by $s \in \mathcal{AB}$). As discussed in Ref. 8, a complete set of generators for the subgroup $G_{\mathcal{A}}$ can be constructed by taking: (i) all star operators that act solely on $\mathcal{A}$, i.e., $\{A_s \mid s \in \mathcal{A}\}$, together with (ii) the collective operators defined as the product of all stars acting solely on a connected component of $\mathcal{B}$ times the product of all boundary stars of that specific component, for all the $m_{\mathcal{B}}$ connected components of $\mathcal{B}$, i.e., $\{\prod_{s\in\mathcal{B}_i} A_s \times \prod_{s'\in\mathcal{AB}_i} A_{s'},\ \forall \text{connected components}\, i\}$. Notice that not all the collective operators are new operators with respect to those generated by the star operators in $\mathcal{A}$. In fact, $\prod_i \Big(\prod_{s'\in\mathcal{B}_i} A_{s'} \prod_{s''\in\mathcal{AB}_i} A_{s''}\Big) \equiv \prod_{s\in\mathcal{A}} A_s$, and one can show that there are precisely $m_{\mathcal{B}} - 1$ new, independent operators. Consequently, the cardinality of the subgroup $G_{\mathcal{A}}$ is given by $d_{\mathcal{A}} \equiv |G_{\mathcal{A}}| = 2^{\Sigma_A + m_B - 1}$. Similarly for $G_{\mathcal{B}}$, $d_{\mathcal{B}} \equiv |G_{\mathcal{B}}| = 2^{\Sigma_B + m_A - 1}$.

To proceed with the calculation of the von Neumann entropy of the finite temperature system, it is useful to use the above set of generators in order to represent the group $G_{\mathcal{A}}$ in terms of Ising spin variables $\{\theta_s, \Theta_i\}_{s\in\mathcal{A}}^{\text{connected components } i}$, where $\theta_s = -1$ (1) corresponds to the star operator A_s appearing (not appearing) in the decomposition of $g \in G_{\mathcal{A}}$, and similarly $\Theta_i = -1$ (1) corresponds to the collective operator $\prod_{s\in\mathcal{B}_i} A_s \times \prod_{s'\in\mathcal{AB}_i} A_{s'}$ appearing (not appearing) in the same decomposition. Notice that the correspondence is 2-to-1, since a configuration $\{\theta_s, \Theta_i\}_{i,s}$ and its spin-flipped

counterpart $\{\overline{\theta}_s, \overline{\Theta}_i\}_{i,s}$, where $\overline{\theta}_s = -\theta_s$ and $\overline{\Theta}_i = -\Theta_i$, map onto the exact same $g \in G_{\mathcal{A}}$ (which follows from the fact that one of the collective operators can be generated out of the others appropriately combined with the star operators in $\mathcal{A}$).

In this representation,

$$\begin{aligned} n(g) &= \sum_s \frac{1-\theta_s(g)}{2} + \sum_i \left(\Sigma_{\mathcal{B}_i} + \Sigma_{\mathcal{AB}_i}\right) \frac{1-\Theta_i(g)}{2} \\ &= \frac{N}{2} - \frac{1}{2}\sum_s \theta_s(g) - \frac{1}{2}\sum_i \Sigma_{\mathcal{P}_i}\Theta_i(g) \qquad (15a) \\ N - n(g) &= \sum_s \frac{1+\theta_s(g)}{2} + \sum_i \left(\Sigma_{\mathcal{B}_i} + \Sigma_{\mathcal{AB}_i}\right) \frac{1+\Theta_i(g)}{2} \\ &= \frac{N}{2} + \frac{1}{2}\sum_s \theta_s(g) + \frac{1}{2}\sum_i \Sigma_{\mathcal{P}_i}\Theta_i(g), \qquad (15b) \end{aligned}$$

where we used the fact that $\Sigma_{\mathcal{A}} + \sum_i \left(\Sigma_{\mathcal{B}_i} + \Sigma_{\mathcal{AB}_i}\right) = N$ and we introduced the notation $\Sigma_{\mathcal{P}_i} \equiv \Sigma_{\mathcal{B}_i} + \Sigma_{\mathcal{AB}_i}$.

Let us then use Eq. (13) to compute the n-th power of $\rho_{\mathcal{A}}(T)$:

$$\begin{aligned} \rho^n_{\mathcal{A}}(T) &= \left(\frac{1}{|G|}\right)^n \sum_{\substack{g_1\in G \\ g'_1 \in G_{\mathcal{A}}}} \cdots \sum_{\substack{g_n\in G \\ g'_n \in G_{\mathcal{A}}}} \left(\prod_{l=1}^n \eta_T(g'_l)\right) \times \\ & g_{1,\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|g_{1,\mathcal{A}}g'_{1,\mathcal{A}}g_{2,\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|g_{2,\mathcal{A}}g'_{2,\mathcal{A}} \\ & \cdots g_{n,\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|g_{n,\mathcal{A}}g'_{n,\mathcal{A}}. \qquad (16) \end{aligned}$$

Each expectation value above imposes $g_{l,\mathcal{A}}g'_{l,\mathcal{A}}g_{l+1,\mathcal{A}} = \mathbf{1}_{\mathcal{A}}$, $l = 1,\ldots,n-1$, and therefore $g_l g'_l g_{l+1} \in G_{\mathcal{B}}$. Upon relabeling $n-1$ summation variables so that $\tilde{g}_{l+1} \equiv g_l g'_l g_{l+1}$ for $l = 1,\ldots,n-1$, the corresponding sums can then be combined with the respective inner product and they can be written as $\sum_{\tilde{g}_{l+1}\in G_{\mathcal{B}}} 1 = d_{\mathcal{B}}$. Therefore, the equation above can be simplified to

$$\begin{aligned} \rho^n_{\mathcal{A}}(T) &= \frac{d^{n-1}_{\mathcal{B}}}{|G|^n} \sum_{g_1\in G} \left(\prod_{l=1}^n \sum_{g'_l\in G_{\mathcal{A}}} \eta_T(g'_l)\right) \times \\ & g_{1,\mathcal{A}}|0_{\mathcal{A}}\rangle\langle 0_{\mathcal{A}}|g_{1,\mathcal{A}}g'_{1,\mathcal{A}} \ \ldots \ g'_{n,\mathcal{A}}. \qquad (17) \end{aligned}$$

Taking the trace of $\rho^n_{\mathcal{A}}(T)$, using the fact that all the g's commute, and $\sum_{g_1\in G} 1 = |G|$, one obtains

$$\begin{aligned} \mathrm{Tr}\left[\rho^n_{\mathcal{A}}\right] &= \\ &= \left(\frac{d_{\mathcal{B}}}{|G|}\right)^{n-1} \prod_{l=1}^n \sum_{g'_l\in G_{\mathcal{A}}} \eta_T(g'_l)\ \langle 0_{\mathcal{A}}|g'_{1,\mathcal{A}} \ \ldots \ g'_{n,\mathcal{A}}|0_{\mathcal{A}}\rangle \\ &= \left(\frac{d_{\mathcal{B}}}{|G|}\right)^{n-1} \frac{1}{2^n} \prod_{l=1}^n \sum^{\text{constr.}}_{\{\theta^{(l)}_s,\Theta^{(l)}_i\}_{i,s}} \eta_T(\{\theta^{(l)}_s,\Theta^{(l)}_i\}), \qquad (18) \end{aligned}$$

where the factor of $1/2^n$ comes from the 2-to-1 nature of the representation of $G_{\mathcal{A}}$ in terms of Ising spin configurations, and the restricted summation $\sum^{\text{constr.}}_{\{\theta^{(l)}_s,\Theta^{(l)}_i\}}$ is subject to the constraint $\langle 0_{\mathcal{A}}|g'_{1,\mathcal{A}}\cdots g'_{n,\mathcal{A}}|0_{\mathcal{A}}\rangle \neq 0$, which can be explicitly stated in terms of the spins θ_s and Θ_i as

$$\begin{aligned} \Bigg\{ & \prod_{i,s} \delta(\prod_{l=1}^n \Theta^{(l)}_i - 1)\delta(\prod_{l=1}^n \theta^{(l)}_s - 1) \\ & + \prod_{i,s} \delta(\prod_{l=1}^n \Theta^{(l)}_i + 1)\delta(\prod_{l=1}^n \theta^{(l)}_s + 1) \Bigg\}. \end{aligned}$$

Above and in the following, the short-hand notation i, s in sums and products stands for connected components i and $s \in \mathcal{A}$.

Let us then substitute Eqs. (15) into Eq. (14),

$$\begin{aligned} & \eta_T(\{\theta^{(l)}_s,\Theta^{(l)}_i\}) = \\ &= \frac{\cosh\left(\frac{K_A}{2}\sum_s \theta^{(l)}_s + \frac{K_A}{2}\sum_i \Sigma_{\mathcal{P}_i}\Theta^{(l)}_i\right)}{\cosh\left(\frac{K_A}{2}N\right)} \\ &= \frac{1}{2\cosh\left(\frac{K_A}{2}N\right)} \times \\ & \qquad \sum_{J=\pm1} e^{\frac{K_A}{2}J\sum_s \theta^{(l)}_s + \frac{K_A}{2}J\sum_i \Sigma_{\mathcal{P}_i}\Theta^{(l)}_i} \\ &= \frac{1}{2\cosh\left(\frac{K_A}{2}N\right)} \times \\ & \qquad \sum_{J=\pm1} \prod_{s\in\mathcal{A}} e^{\frac{K_A}{2}J\theta^{(l)}_s} \prod_i e^{\frac{K_A}{2}J\Sigma_{\mathcal{P}_i}\Theta^{(l)}_i}, \qquad (19) \end{aligned}$$

and take the sum over all possible $\{\theta^{(l)}_s,\Theta^{(l)}_i\}_{i,s}$ configurations (without any constraint),

$$\begin{aligned} & \left[2\cosh\left(\frac{K_A}{2}N\right)\right] \sum_{\{\theta^{(l)}_s,\Theta^{(l)}_i\}_{i,s}} \eta_T(\{\theta^{(l)}_s,\Theta^{(l)}_i\}) = \\ &= \sum_{J=\pm1} \sum_{\{\theta^{(l)}_s,\Theta^{(l)}_i\}_{i,s}} \prod_{s\in\mathcal{A}} e^{\frac{K_A}{2}J\theta^{(l)}_s} \prod_i e^{\frac{K_A}{2}J\Sigma_{\mathcal{P}_i}\Theta^{(l)}_i} \\ &= \sum_{J=\pm1} \left(\prod_s \sum_{\theta^{(l)}_s=\pm1} e^{\frac{K_A}{2}J\theta^{(l)}_s}\right) \times \\ & \qquad \left(\prod_i \sum_{\Theta^{(l)}_i=\pm1} e^{\frac{K_A}{2}J\Sigma_{\mathcal{P}_i}\Theta^{(l)}_i}\right). \qquad (20) \end{aligned}$$

Using the expression above, one can rewrite Eq. 18 as

$$\mathrm{Tr}\,[\rho_{\mathcal{A}}^n] \;=\; \left(\frac{d_{\mathcal{B}}}{|G|}\right)^{n-1} \frac{1}{2^{2n}\left[\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)\right]^n} \prod_{l=1}^{n}\sum_{J_l=\pm 1}\left(\prod_s \sum_{\theta_s^{(l)}=\pm 1} e^{\frac{K_{\mathcal{A}}}{2}J_l\,\theta_s^{(l)}}\right)\left(\prod_i \sum_{\Theta_i^{(l)}=\pm 1} e^{\frac{K_{\mathcal{A}}}{2}J_l\,\Sigma_{\mathcal{P}_i}\Theta_i^{(l)}}\right)\times$$
$$\left\{\prod_{i,s}\delta(\prod_{l=1}^{n}\Theta_i^{(l)}-1)\delta(\prod_{l=1}^{n}\theta_s^{(l)}-1)+\prod_{i,s}\delta(\prod_{l=1}^{n}\Theta_i^{(l)}+1)\delta(\prod_{l=1}^{n}\theta_s^{(l)}+1)\right\}$$

and, upon expanding the product $\prod_{l=1}^{n}\left(\sum_{J_l=\pm 1}C(J_l)\right)=\sum_{\{J_l\}_{l=1}^n}\prod_{l=1}^{n}C(J_l)$, we obtain

$$\mathrm{Tr}\,[\rho_{\mathcal{A}}^n] \;=\; \left(\frac{d_{\mathcal{B}}}{|G|}\right)^{n-1} \frac{1}{2^{2n}\left[\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)\right]^n} \sum_{\{J_l\}_{l=1}^n}\left\{\left(\prod_s \sum_{\substack{\{\theta_s^{(l)}\}_{l=1}^n\\ \prod\theta_s^{(l)}=+1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_l J_l\,\theta_s^{(l)}}\right)\left(\prod_i \sum_{\substack{\{\Theta_i^{(l)}\}_{l=1}^n\\ \prod\Theta_i^{(l)}=+1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_l J_l\,\Sigma_{\mathcal{P}_i}\Theta_i^{(l)}}\right)+\right.$$
$$\left.\left(\prod_s \sum_{\substack{\{\theta_s^{(l)}\}_{l=1}^n\\ \prod\theta_s^{(l)}=-1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_l J_l\,\theta_s^{(l)}}\right)\left(\prod_i \sum_{\substack{\{\Theta_i^{(l)}\}_{l=1}^n\\ \prod\Theta_i^{(l)}=+1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_l J_l\,\Sigma_{\mathcal{P}_i}\Theta_i^{(l)}}\right)\right\} \tag{21}$$

Notice that the indices s and i to the variables $\theta_s^{(l)}$ and $\Theta_i^{(l)}$, respectively, are mute since the sums over the possible values of $\theta_s^{(l)}$ and $\Theta_i^{(l)}$ are performed first. Therefore one can simplify the notation above by replacing $\theta_s^{(l)}\to\theta^{(l)}$ and $\Theta_i^{(l)}\to\Theta^{(l)}$ for all s and i. Moreover, one can recognize that an Ising spin chain $\{\theta^{(l)}\}$ with $\prod\theta^{(l)}=\pm 1$ is dual to an Ising spin chain $\{\tau_l\}$ with periodic / antiperiodic boundary conditions (using the 2-to-1 mapping $\theta^{(l)}=\tau_l\,\tau_{l+1}$). Thus,

$$\sum_{\substack{\{\theta^{(l)}\}_{l=1}^n\\ \prod\theta^{(l)}=\pm 1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_{l=1}^n J_l\,\theta^{(l)}} \;=\; \frac{1}{2}\sum_{\{\tau_l\}_{l=1}^n}^{\text{p./a.}} e^{\frac{K_{\mathcal{A}}}{2}\sum_{l=1}^n J_l\,\tau_l\,\tau_{l+1}}$$
$$\equiv\; \frac{1}{2}Z_n^{(p/a)}(K_A,\{J_l\}), \tag{22}$$

where $Z_n^{(p/a)}(K_A,\{J_l\})$ is the partition function of a chain of n Ising spins with periodic / antiperiodic boundary conditions, in presence of a nearest-neighbor interaction with position-dependent reduced coupling constant $K_A J_l/2$. Similarly,

$$\sum_{\substack{\{\Theta^{(l)}\}_{l=1}^n\\ \prod\Theta^{(l)}=\pm 1}} e^{\frac{K_{\mathcal{A}}}{2}\sum_{l=1}^n \Sigma_{\mathcal{P}_i}J_l\,\Theta^{(l)}} \;\equiv\; \frac{1}{2}Z_n^{(p/a)}(K_A\Sigma_{\mathcal{P}_i},\{J_l\}), \tag{23}$$

and Eq. (21) can be rewritten as

$$\mathrm{Tr}\,[\rho_{\mathcal{A}}^n] \;=\; \frac{1}{2^{\Sigma_{\mathcal{A}}+m_{\mathcal{B}}}}\left(\frac{d_{\mathcal{B}}}{|G|}\right)^{n-1} \frac{1}{2^{2n}\left[\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)\right]^n}\sum_{\{J_l\}_{l=1}^n}\times$$
$$\left\{\left(Z_n^{(p)}(K_A,\{J_l\})\right)^{\Sigma_{\mathcal{A}}}\prod_i\left(Z_n^{(p)}(K_A\Sigma_{\mathcal{P}_i},\{J_l\})\right)+\left(Z_n^{(a)}(K_A,\{J_l\})\right)^{\Sigma_{\mathcal{A}}}\prod_i\left(Z_n^{(a)}(K_A\Sigma_{\mathcal{P}_i},\{J_l\})\right)\right\}. \tag{24}$$

The partition functions can be evaluated, for either boundary conditions, using a transfer matrix approach. For periodic boundary conditions, we obtain

$$Z_n^{(p)}(K,\{J_l\})=\mathrm{Tr}\left[\prod_{l=1}^{n}T_l\right],\qquad T_l\;=\begin{pmatrix} e^{\frac{K}{2}J_l} & e^{-\frac{K}{2}J_l}\\ e^{-\frac{K}{2}J_l} & e^{\frac{K}{2}J_l}\end{pmatrix}.$$

Since all matrices T_l are diagonalized by the same unitary matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{25}$$

and using the cyclic properties of the trace, we get

$$\begin{aligned} Z_n^{(p)}(K,\{J_l\}) &= \mathrm{Tr}\left[\prod_{l=1}^{n}(UT_lU^\dagger)\right] \\ &= \mathrm{Tr}\left[\prod_{l=1}^{n}\begin{pmatrix} e^{\frac{K}{2}J_l}+e^{-\frac{K}{2}J_l} & 0 \\ 0 & e^{\frac{K}{2}J_l}-e^{-\frac{K}{2}J_l} \end{pmatrix}\right] \\ &= \mathrm{Tr}\left[\prod_{l=1}^{n}\begin{pmatrix} 2\cosh(\frac{K}{2}) & 0 \\ 0 & 2J_l\sinh(\frac{K}{2}) \end{pmatrix}\right] \\ &= \left[2\cosh\left(\frac{K}{2}\right)\right]^n + \left(\prod_{l=1}^{n} J_l\right)\left[2\sinh\left(\frac{K}{2}\right)\right]^n . \end{aligned} \tag{26}$$

For the antiperiodic case,

$$Z_n^{(a)}(K,\{J_l\}) = \mathrm{Tr}\left[\sigma_1 \prod_{l=1}^{n} T_l\right] \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and we get

$$\begin{aligned} Z_n^{(a)}(K,\{J_l\}) &= \mathrm{Tr}\left[(U\sigma_1U^\dagger)\prod_{l=1}^{n}(UT_lU^\dagger)\right] \\ &= \left[2\cosh\left(\frac{K}{2}\right)\right]^n - \left(\prod_{l=1}^{n} J_l\right)\left[2\sinh\left(\frac{K}{2}\right)\right]^n . \end{aligned} \tag{27}$$

We can substitute into Eq. (24) and we obtain

$$\begin{aligned} \mathrm{Tr}\left[\rho_{\mathcal{A}}^n\right] &= \left(\frac{d_{\mathcal{B}}\, 2^{\Sigma_{\mathcal{A}}+m_{\mathcal{B}}}}{|G|}\right)^{n-1} \frac{1}{2^{2n}\left[\cosh\left(\frac{K_A}{2}N\right)\right]^n} \sum_{\{J_l\}_{l=1}^n} \times \\ &\Bigg\{\left(\left[\cosh\left(\frac{K_A}{2}\right)\right]^n + \left(\prod_{l=1}^{n} J_l\right)\left[\sinh\left(\frac{K_A}{2}\right)\right]^n\right)^{\Sigma_{\mathcal{A}}} \prod_i \left(\left[\cosh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n + \left(\prod_{l=1}^{n} J_l\right)\left[\sinh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n\right) \\ &+ \left(\left[\cosh\left(\frac{K_A}{2}\right)\right]^n - \left(\prod_{l=1}^{n} J_l\right)\left[\sinh\left(\frac{K_A}{2}\right)\right]^n\right)^{\Sigma_{\mathcal{A}}} \prod_i \left(\left[\cosh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n - \left(\prod_{l=1}^{n} J_l\right)\left[\sinh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n\right)\Bigg\}. \end{aligned}$$

Notice that the factors $\prod_l J_l$ can be dropped since the two terms between curly brackets in the equation above get simply exchanged by the two different values of $\prod_l J_l = \pm 1$. The summation over $\{J_l\}_{l=1}^n$ gives therefore just a multiplicative factor 2^n. Recalling the definition $d_{\mathcal{A}} = 2^{\Sigma_{\mathcal{A}}+m_{\mathcal{B}}-1}$, we finally get to

$$\begin{aligned} \mathrm{Tr}\left[\rho_{\mathcal{A}}^n\right] &= \frac{1}{2\cosh\left(\frac{K_A}{2}N\right)}\left(\frac{1}{\cosh\left(\frac{K_A}{2}N\right)}\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}\right)^{n-1} \times \\ &\Bigg\{\left(\left[\cosh\left(\frac{K_A}{2}\right)\right]^n + \left[\sinh\left(\frac{K_A}{2}\right)\right]^n\right)^{\Sigma_{\mathcal{A}}} \prod_i \left(\left[\cosh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n + \left[\sinh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n\right) + \\ &\left(\left[\cosh\left(\frac{K_A}{2}\right)\right]^n - \left[\sinh\left(\frac{K_A}{2}\right)\right]^n\right)^{\Sigma_{\mathcal{A}}} \prod_i \left(\left[\cosh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n - \left[\sinh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right)\right]^n\right)\Bigg\}. \end{aligned} \tag{28}$$

We can now use the identity

$$\mathrm{Tr}\left[\rho_{\mathcal{A}}\ln\rho_{\mathcal{A}}\right] = \lim_{n\to 1}\partial_n \mathrm{Tr}\left[\rho_{\mathcal{A}}^n\right] \tag{29}$$

to compute the von Neumann entropy $S_{VN}^{\mathcal{A}}(T) = -\mathrm{Tr}\left[\rho_{\mathcal{A}}\ln\rho_{\mathcal{A}}\right]$. It is convenient to introduce the following simplified notation:

$$x = \cosh\left(\frac{K_A}{2}\right) \qquad y = \sinh\left(\frac{K_A}{2}\right) \tag{30a}$$

$$\tilde{x}_i = \cosh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right) \qquad \tilde{y}_i = \sinh\left(\frac{K_A}{2}\Sigma_{\mathcal{P}_i}\right), \tag{30b}$$

which allows us to rewrite the terms in the last two lines of Eq. (28) as

$$F_+^{(n)} = (x^n + y^n)^{\Sigma_\mathcal{A}} \prod_i (\tilde{x}_i^n + \tilde{y}_i^n) \tag{31}$$

$$F_-^{(n)} = (x^n - y^n)^{\Sigma_\mathcal{A}} \prod_i (\tilde{x}_i^n - \tilde{y}_i^n) . \tag{32}$$

One can verify that

$$\begin{aligned} F_\pm^{(1)} &= e^{\pm \frac{K_\mathcal{A}}{2}\left(\Sigma_\mathcal{A} + \sum_i \Sigma_{\mathcal{P}_i}\right)} \\ &= e^{\pm \frac{K_\mathcal{A}}{2} N} \end{aligned} \tag{33}$$

and

$$\begin{aligned} \partial_n F_\pm^{(n)}\Big|_{n=1} &= \Sigma_\mathcal{A} e^{\pm \frac{K_\mathcal{A}}{2}(N-1)} \left(x \ln x \pm y \ln y\right) \\ &\quad + \sum_i e^{\pm \frac{K_\mathcal{A}}{2}\left(N - \Sigma_{\mathcal{P}_i}\right)} \left(\tilde{x}_i \ln \tilde{x}_i \pm \tilde{y}_i \ln \tilde{y}_i\right) . \end{aligned} \tag{34}$$

Using this notation

$$\mathrm{Tr}\left[\rho_\mathcal{A}^n\right] = \frac{1}{2\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \left(\frac{1}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \frac{d_\mathcal{A} d_\mathcal{B}}{|G|}\right)^{n-1} \left\{F_+^{(n)} + F_-^{(n)}\right\} .$$

and

$$\begin{aligned} S_{\mathrm{VN}}^{\mathcal{A}}(T) &= -\frac{1}{2\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \left\{ \ln\left[\frac{1}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \frac{d_\mathcal{A} d_\mathcal{B}}{|G|}\right] 2\cosh\left(\frac{K_A}{2}N\right) \right. \\ &\qquad + \Sigma_\mathcal{A} e^{\frac{K_\mathcal{A}}{2}(N-1)} \left(x \ln x + y \ln y\right) + \sum_i e^{\frac{K_\mathcal{A}}{2}\left(N - \Sigma_{\mathcal{P}_i}\right)} \left(\tilde{x}_i \ln \tilde{x}_i + \tilde{y}_i \ln \tilde{y}_i\right) \\ &\qquad \left. + \Sigma_\mathcal{A} e^{-\frac{K_\mathcal{A}}{2}(N-1)} \left(x \ln x - y \ln y\right) + \sum_i e^{-\frac{K_\mathcal{A}}{2}\left(N - \Sigma_{\mathcal{P}_i}\right)} \left(\tilde{x}_i \ln \tilde{x}_i - \tilde{y}_i \ln \tilde{y}_i\right) \right\} \\ &= -\ln \frac{d_\mathcal{A} d_\mathcal{B}}{|G|} + \ln\cosh\left(\frac{K_A}{2}N\right) \\ &\quad - \frac{1}{2\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \left\{ \Sigma_\mathcal{A} e^{\frac{K_\mathcal{A}}{2}(N-1)} \left(x \ln x + y \ln y\right) + \sum_i e^{\frac{K_\mathcal{A}}{2}\left(N - \Sigma_{\mathcal{P}_i}\right)} \left(\tilde{x}_i \ln \tilde{x}_i + \tilde{y}_i \ln \tilde{y}_i\right) \right. \\ &\quad \left. + \Sigma_\mathcal{A} e^{-\frac{K_\mathcal{A}}{2}(N-1)} \left(x \ln x - y \ln y\right) + \sum_i e^{-\frac{K_\mathcal{A}}{2}\left(N - \Sigma_{\mathcal{P}_i}\right)} \left(\tilde{x}_i \ln \tilde{x}_i - \tilde{y}_i \ln \tilde{y}_i\right) \right\} \\ &= -\ln \frac{d_\mathcal{A} d_\mathcal{B}}{|G|} + \ln\cosh\left(\frac{K_A}{2}N\right) - \Sigma_\mathcal{A} \left(x \ln x\right) \frac{\cosh\left(\frac{K_\mathcal{A}}{2}(N-1)\right)}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} - \Sigma_\mathcal{A} \left(y \ln y\right) \frac{\sinh\left(\frac{K_\mathcal{A}}{2}(N-1)\right)}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \\ &\quad - \sum_i \left(\tilde{x}_i \ln \tilde{x}_i\right) \frac{\cosh\left(\frac{K_\mathcal{A}}{2}(N - \Sigma_{\mathcal{P}_i})\right)}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} - \sum_i \left(\tilde{y}_i \ln \tilde{y}_i\right) \frac{\sinh\left(\frac{K_\mathcal{A}}{2}(N - \Sigma_{\mathcal{P}_i})\right)}{\cosh\left(\frac{K_\mathcal{A}}{2}N\right)} \end{aligned} \tag{35}$$

For a finite size system, the limit of $K_A \to 0^+$ yields $x, \tilde{x}_i \to 1$, $y, \tilde{y}_i \to 0$ and therefore

$$S_{\mathrm{VN}}^{\mathcal{A}}(T \to 0) \to -\ln \frac{d_\mathcal{A} d_\mathcal{B}}{|G|}, \tag{36}$$

consistently with the known zero-temperature result.[13]

In the limit of $K_A \to \infty$ instead, $x, y \sim e^{K_A/2}/2$, $\tilde{x}_i, \tilde{y}_i \sim$

$e^{K_A \Sigma_{\mathcal{P}_i}/2}/2$, and therefore

$$
\begin{aligned}
& S_{\rm VN}^{\mathcal{A}}(T \to \infty) \to \\
& \quad \to -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} + \frac{K_A}{2} N - \ln 2 - \Sigma_{\mathcal{A}} \left(\frac{K_A}{2} - \ln 2 \right) \\
& \qquad - \sum_i \left(\frac{K_A}{2} \Sigma_{\mathcal{P}_i} - \ln 2 \right) \\
& \quad = -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} - \ln 2 + \Sigma_{\mathcal{A}} \ln 2 + m_{\mathcal{B}} \ln 2 \\
& \quad = -\ln \frac{d_{\mathcal{B}}}{|G|}, \qquad (37)
\end{aligned}
$$

where $d_{\mathcal{A}} = 2^{\Sigma_{\mathcal{A}} + m_{\mathcal{B}} - 1}$. This result is consistent with Ref. 8, where the classical system that obtains in the $T \to \infty$ limit had been previously discussed.

Let us then consider the thermodynamic limit $L \to \infty$. The presence of terms that depend on the product between K_A and extensive quantities such as N, $\Sigma_{\mathcal{A}}$, and $\Sigma_{\mathcal{P}_i}$ requires careful consideration. As we will see, they will indeed lead to a singular behavior of the von Neumann entropy, as well as the topological entropy discussed in Sec. IV. For any finite $K_A \in (0, \infty)$, the limit $L \to \infty$ ($N = L^2 \to \infty$) yields

$$
\begin{aligned}
& S_{\rm VN}^{\mathcal{A}}(T) \overset{L \to \infty}{\longrightarrow} \\
& \quad \longrightarrow -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} + \frac{K_A}{2} N - \ln 2 \\
& \qquad - \Sigma_{\mathcal{A}} \left(x \ln x + y \ln y \right) e^{-\frac{K_A}{2}} \\
& \qquad - \sum_i \left(\tilde{x}_i \ln \tilde{x}_i + \tilde{y}_i \ln \tilde{y}_i \right) e^{-\frac{K_A}{2} \Sigma_{\mathcal{P}_i}}. \qquad (38)
\end{aligned}
$$

We can further simplify this expression in the limit $\Sigma_{\mathcal{P}_i}, \Sigma_{\mathcal{A}} \gg 1$, i.e., in the limit of large partitions;[15] this is the case of interest, for example, in the definition of the topological entropy discussed in the Section IV. If we expand $\tilde{x}_i, \tilde{y}_i \sim e^{K_A \Sigma_{\mathcal{P}_i}/2}/2$, we obtain

$$
\begin{aligned}
& S_{\rm VN}^{\mathcal{A}}(T) \overset{L \to \infty}{\longrightarrow} \\
& \quad \longrightarrow -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} + \frac{K_A}{2} N - \ln 2 \\
& \qquad - \Sigma_{\mathcal{A}} e^{-\frac{K_A}{2}} \left(x \ln x + y \ln y \right) \\
& \qquad - \sum_i \left(\frac{K_A}{2} \Sigma_{\mathcal{P}_i} - \ln 2 \right) \\
& \quad = -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} + (m_{\mathcal{B}} - 1) \ln 2 \\
& \qquad - \Sigma_{\mathcal{A}} \left[e^{-\frac{K_A}{2}} \left(x \ln x + y \ln y \right) - \frac{K_A}{2} \right] \\
& \quad = \Big(\Sigma_{\mathcal{AB}} - m_{\mathcal{A}} \Big) \ln 2 \\
& \qquad - \Sigma_{\mathcal{A}} \left[e^{-\frac{K_A}{2}} \left(x \ln x + y \ln y \right) - \frac{K_A}{2} \right], \qquad (39)
\end{aligned}
$$

which is consistent with the limit $S_{\rm VN}^{\mathcal{A}}(T \to \infty) = -\ln(d_{\mathcal{B}}/|G|)$, but *no longer consistent* with the finite size result $S_{\rm VN}^{\mathcal{A}}(T \to 0) = -\ln(d_{\mathcal{A}} d_{\mathcal{B}}/|G|)$. In fact,

$$
\begin{aligned}
\lim_{T \to 0, L \to \infty} S_{\rm VN}^{\mathcal{A}}(T) &= -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|} + (m_{\mathcal{B}} - 1) \ln 2, \\
&= \Big(\Sigma_{\mathcal{AB}} - m_{\mathcal{A}} \Big) \ln 2 \qquad (40)
\end{aligned}
$$

while

$$
\begin{aligned}
\lim_{L \to \infty, T \to 0} S_{\rm VN}^{\mathcal{A}}(T) &= -\ln \frac{d_{\mathcal{A}} d_{\mathcal{B}}}{|G|}. \\
&= \Big(\Sigma_{\mathcal{AB}} - m_{\mathcal{A}} - m_{\mathcal{B}} + 1 \Big) \ln 2 \qquad (41)
\end{aligned}
$$

Notice that already in Eq. (38) the limit $T \to 0$ fails to give the known result, Eq. (36). However, it is only in the assumption that all partitions scale with the system size that we arrive at Eq. (39). Notice also that the difference between the two orders of limits arises only if subsystem $\mathcal{B}$ has more than one connected component. Moreover, such difference is of order one, while the common term $-\ln(d_{\mathcal{A}} d_{\mathcal{B}})/|G|$ scales linearly with the size of the boundary between $\mathcal{A}$ and $\mathcal{B}$ ($\propto \Sigma_{\mathcal{AB}}$). Therefore, from a thermodynamic point of view the order of limits is immaterial to the von Neumann entanglement entropy. On the other hand, we will see in the following section how this difference plays a crucial role in the topological entropy of the system, and gives rise to a singular behavior at zero temperature in the thermodynamic limit.

A. The mutual information

In the previous section, and in particular from Eq. (35) we clearly see that the von Neumann entropy is symmetric upon exchange of $\mathcal{A}$ and $\mathcal{B}$, and it satisfies the so-called area law only at zero temperature (and for topologically ordered systems, symmetry is lost unless the $T \to 0$ limit is taken before the $L \to \infty$ limit). At any finite temperature, $S_{\rm VN}^{\mathcal{A}}(T)$ acquires an extensive contribution scaling with the number of degrees of freedom in partition $\mathcal{A}$. In this sense, the von Neumann entropy ceases to be a good measure of the entropy contained in the boundary between the two subsystems, and therefore is no longer a good measure of entanglement at finite temperature.

Alternatively, one can consider another quantity called the mutual information $I_{\mathcal{AB}}$, which we redefine for convenience multiplied by a factor $1/2$,

$$
I_{\mathcal{AB}}(T) = \frac{1}{2} \left[S_{\rm VN}^{\mathcal{A}}(T) + S_{\rm VN}^{\mathcal{B}}(T) - S_{\rm VN}^{\mathcal{A} \cup \mathcal{B}}(T) \right], \qquad (42)
$$

so that $I_{\mathcal{AB}}(0, N) = S_{\rm VN}^{\mathcal{A}}(0, N) = S_{\rm VN}^{\mathcal{B}}(0, N)$. Substituting the result from Eq. (35), and recalling that $d_{\mathcal{A} \cup \mathcal{B}} = |G|$, $d_{\emptyset} = 1$ and $\Sigma_{\mathcal{A} \cup \mathcal{B}} = N$, we obtain the behavior of $I_{\mathcal{AB}}(T)$ as a function of temperature and system size for a topologically ordered system,

$$
\begin{aligned}
I_{\mathcal{AB}}(T) \;=\; & -\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}+\frac{1}{2}\ln\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)+\frac{\Sigma_{\mathcal{AB}}}{2}\left\{(x\ln x)\,\frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N-1)\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}+(y\ln y)\,\frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N-1)\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}\right\}\\
& -\frac{1}{2}\sum_i(\tilde{x}_i\ln\tilde{x}_i)\,\frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N-\Sigma_{\mathcal{B}_i}-\Sigma_{\mathcal{AB}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}-\frac{1}{2}\sum_i(\tilde{y}_i\ln\tilde{y}_i)\,\frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N-\Sigma_{\mathcal{B}_i}-\Sigma_{\mathcal{AB}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}\\
& -\frac{1}{2}\sum_i(\tilde{x}_i\ln\tilde{x}_i)\,\frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N-\Sigma_{\mathcal{A}_i}-\Sigma_{\mathcal{AB}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}-\frac{1}{2}\sum_i(\tilde{y}_i\ln\tilde{y}_i)\,\frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N-\Sigma_{\mathcal{A}_i}-\Sigma_{\mathcal{AB}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}. \qquad (43)
\end{aligned}
$$

The mutual information has the immediate advantage over the von Neumann entropy that it is symmetric upon exchange of subsystem $\mathcal{A}$ with $\mathcal{B}$. Moreover, while the general expression above seems to have an explicit dependence on the bulk degrees of freedom ($\Sigma_{\mathcal{A}_i}$ and $\Sigma_{\mathcal{B}_i}$), we show hereafter that in the thermodynamic limit as well as in the zero-temperature limit the mutual information depends only on the boundary degrees of freedom ($\Sigma_{\mathcal{AB}}$) and, as such, is a good candidate for a measure of entanglement at finite as well as zero temperature.

In the $T\to 0$ limit we recover the known result

$$
\begin{aligned}
I_{\mathcal{AB}}(0) &= -\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}\\
&= \left(\Sigma_{\mathcal{AB}}-m_{\mathcal{A}}-m_{\mathcal{B}}+1\right)\ln 2. \qquad (44)
\end{aligned}
$$

In the $T\to\infty$ limit instead we obtain

$$
\begin{aligned}
I_{\mathcal{AB}}(\infty) &= -\frac{1}{2}\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}\\
&= \frac{1}{2}\left(\Sigma_{\mathcal{AB}}-m_{\mathcal{A}}-m_{\mathcal{B}}+1\right)\ln 2, \qquad (45)
\end{aligned}
$$

which again shows only a dependence on boundary degrees of freedom, with the addition of topological terms of order one, and is symmetric under the exchange of $\mathcal{A}$ and $\mathcal{B}$.

We can then compare these finite-size results with the behavior of $I_{\mathcal{AB}}(T)$ in the thermodynamic limit $L\to\infty$ for large partitions,

$$
\begin{aligned}
I_{\mathcal{AB}}(T) \simeq\; & -\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}\\
& +\frac{\Sigma_{\mathcal{AB}}}{2}\left\{(x\ln x+y\ln y)\,e^{-\frac{K_{\mathcal{A}}}{2}}-\frac{K_{\mathcal{A}}}{2}\right\}\\
& +\frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\ln 2\\
=\; & \frac{\Sigma_{\mathcal{AB}}}{2}\left\{2\ln 2+(x\ln x+y\ln y)\,e^{-\frac{K_{\mathcal{A}}}{2}}-\frac{K_{\mathcal{A}}}{2}\right\}\\
& -\frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\ln 2. \qquad (46)
\end{aligned}
$$

Once again this is consistent with the finite size limit $T\to\infty$, but *not* with the $T\to 0$ limit. In fact,

$$
\begin{aligned}
\lim_{L\to\infty,T\to 0} I_{\mathcal{AB}}(T) &= -\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}\\
&= \left(\Sigma_{\mathcal{AB}}-m_{\mathcal{A}}-m_{\mathcal{B}}+1\right)\ln 2 \qquad (47)
\end{aligned}
$$

while

$$
\begin{aligned}
\lim_{T\to 0,L\to\infty} I_{\mathcal{AB}}(T) &= -\ln\frac{d_{\mathcal{A}}d_{\mathcal{B}}}{|G|}+\frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\ln 2\\
&= \left[\Sigma_{\mathcal{AB}}-\frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\right]\ln 2. \qquad (48)
\end{aligned}
$$

The difference between the mutual information obtained for the two order of limits gives

$$
\begin{aligned}
\Delta I_{\mathcal{AB}} &= \lim_{T\to 0,L\to\infty} I_{\mathcal{AB}}(T)-\lim_{L\to\infty,T\to 0} I_{\mathcal{AB}}(T)\\
&= \frac{1}{2}\left(m_{\mathcal{A}}+m_{\mathcal{B}}-1\right)\ln 2. \qquad (49)
\end{aligned}
$$

It is noteworthy that the boundary contribution drops out from $\Delta I_{\mathcal{AB}}$, but that this quantity still has an $\mathcal{O}(1)$ topological contribution that depends on the total number of disconnected regions $m_{\mathcal{A}}+m_{\mathcal{B}}$ of partitions $\mathcal{A}$ and $\mathcal{B}$. Hence a topological contribution to the entanglement entropy can be filtered out directly from a single bipartition using $\Delta I_{\mathcal{AB}}$, in contrast to the constructions in Refs. 5,6 that require a linear combination over multiple bipartitions.

IV. THE TOPOLOGICAL ENTROPY

Let us now compute the topological entropy using the results for the von Neumann entropy in the previous section, and the definition given by Levin and Wen[5]

$$
S_{\text{topo}}=\lim_{r,R\to\infty}\left[-S_{\text{VN}}^{1\mathcal{A}}+S_{\text{VN}}^{2\mathcal{A}}+S_{\text{VN}}^{3\mathcal{A}}-S_{\text{VN}}^{4\mathcal{A}}\right] \qquad (50)
$$

based on the bipartitions shown in Fig. 3. In this specific case, $m_{1B}=2$, with two distinct $\Sigma_{1\mathcal{P}_1}$ and $\Sigma_{1\mathcal{P}_2}$ and $m_{2B}=m_{3B}=m_{4B}=1$, with a single $\Sigma_{2\mathcal{P}}$, $\Sigma_{3\mathcal{P}}$, $\Sigma_{4\mathcal{P}}$, respectively.

From Eq. (35), it follows that

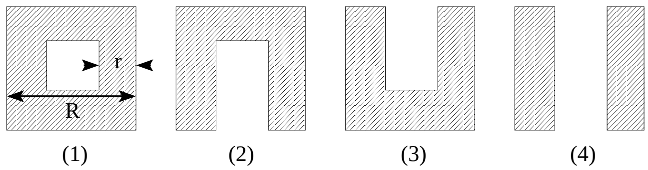

FIG. 3: Illustration of the four bipartitions used to compute the topological entropy in Ref. 5.

$$
\begin{aligned}
S_{\text{topo}}(T) - S_{\text{topo}}^{\text{Kitaev}} &= \sum_{i=1}^{2}\left(\tilde{x}_i^{(1)} \ln \tilde{x}_i^{(1)}\right) \frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{1\mathcal{P}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} + \sum_{i=1}^{2}\left(\tilde{y}_i^{(1)} \ln \tilde{y}_i^{(1)}\right) \frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{1\mathcal{P}_i})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} \\
&- \left(\tilde{x}^{(2)} \ln \tilde{x}^{(2)}\right) \frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{2\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} - \left(\tilde{y}^{(2)} \ln \tilde{y}^{(2)}\right) \frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{2\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} \\
&- \left(\tilde{x}^{(3)} \ln \tilde{x}^{(3)}\right) \frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{3\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} - \left(\tilde{y}^{(3)} \ln \tilde{y}^{(3)}\right) \frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{3\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} \\
&+ \left(\tilde{x}^{(4)} \ln \tilde{x}^{(4)}\right) \frac{\cosh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{4\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)} + \left(\tilde{y}^{(4)} \ln \tilde{y}^{(4)}\right) \frac{\sinh\left(\frac{K_{\mathcal{A}}}{2}(N - \Sigma_{4\mathcal{P}})\right)}{\cosh\left(\frac{K_{\mathcal{A}}}{2}N\right)}, \qquad (51)
\end{aligned}
$$

where $S_{\text{topo}}^{\text{Kitaev}}$ is the topological entropy of the GS of the original toric code, which obtains from the $-\ln(d_{\mathcal{A}} d_{\mathcal{B}})/|G|$ contribution to $S_{\text{VN}}^{\mathcal{A}}(T)$. All the contributions from the $(x \ln x)$ and $(y \ln y)$ terms cancel since $\Sigma_{1\mathcal{A}} - \Sigma_{2\mathcal{A}} - \Sigma_{3\mathcal{A}} + \Sigma_{4\mathcal{A}} = 0$ by construction.

For finite systems, the known limiting values are recovered:

$$
\begin{aligned}
S_{\text{topo}}(T \to 0) - S_{\text{topo}}^{\text{Kitaev}} &\to 0 \qquad (52) \\
S_{\text{topo}}(T \to \infty) - S_{\text{topo}}^{\text{Kitaev}} &\to -\ln 2. \qquad (53)
\end{aligned}
$$

In the thermodynamic limit $L \to \infty$ and with r, R kept constant, all $\Sigma_{\alpha\mathcal{P}_i}$ diverge with the exception of $\Sigma_{1\mathcal{P}_1}$, which corresponds to the inner square of size $(R-2r)^2$ in Fig. 3. Thus, Eq. (51) becomes

$$
\begin{aligned}
S_{\text{topo}}(T) - S_{\text{topo}}^{\text{Kitaev}} &= \sum_{i=1}^{2}\left(\tilde{x}_i^{(1)} \ln \tilde{x}_i^{(1)} + \tilde{y}_i^{(1)} \ln \tilde{y}_i^{(1)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{1\mathcal{P}_i}} - \left(\tilde{x}^{(2)} \ln \tilde{x}^{(2)} + \tilde{y}^{(2)} \ln \tilde{y}^{(2)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{2\mathcal{P}}} \\
&- \left(\tilde{x}^{(3)} \ln \tilde{x}^{(3)} + \tilde{y}^{(3)} \ln \tilde{y}^{(3)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{3\mathcal{P}}} + \left(\tilde{x}^{(4)} \ln \tilde{x}^{(4)} + \tilde{y}^{(4)} \ln \tilde{y}^{(4)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{4\mathcal{P}}} \\
&= \left(\tilde{x}_1^{(1)} \ln \tilde{x}_1^{(1)} + \tilde{y}_1^{(1)} \ln \tilde{y}_1^{(1)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{1\mathcal{P}_1}} + \frac{K_A}{2}\Sigma_{1\mathcal{P}_2} - \frac{K_A}{2}\Sigma_{2\mathcal{P}} - \frac{K_A}{2}\Sigma_{3\mathcal{P}} + \frac{K_A}{2}\Sigma_{4\mathcal{P}} \\
&= \left(\tilde{x}_1^{(1)} \ln \tilde{x}_1^{(1)} + \tilde{y}_1^{(1)} \ln \tilde{y}_1^{(1)}\right) e^{-\frac{K_{\mathcal{A}}}{2}\Sigma_{1\mathcal{P}_1}} - \frac{K_A}{2}\Sigma_{1\mathcal{P}_1}, \qquad (54)
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
N &= \Sigma_{1\mathcal{A}} + \Sigma_{1\mathcal{P}_1} + \Sigma_{1\mathcal{P}_2} \qquad (55) \\
&= \Sigma_{2\mathcal{A}} + \Sigma_{2\mathcal{P}} \qquad (56) \\
&= \Sigma_{3\mathcal{A}} + \Sigma_{3\mathcal{P}} \qquad (57) \\
&= \Sigma_{4\mathcal{A}_1} + \Sigma_{4\mathcal{A}_2} + \Sigma_{4\mathcal{P}}, \qquad (58)
\end{aligned}
$$

and that

$$
\Sigma_{1\mathcal{A}} - \Sigma_{2\mathcal{A}} - \Sigma_{3\mathcal{A}} + \Sigma_{4\mathcal{A}} = 0 \qquad (59)
$$

to substitute

$$\Sigma_{1\mathcal{P}_2} - \Sigma_{2\mathcal{P}} - \Sigma_{3\mathcal{P}} + \Sigma_{4\mathcal{P}} = -\Sigma_{1\mathcal{P}_1} \tag{60}$$

into Eq.(54). Considering that we are eventually interested in taking limit $r, R \to \infty$ (i.e., $\Sigma_{1\mathcal{P}_1} \gg 1$), we obtain the asymptotic value

$$S_{\text{topo}}(T) - S_{\text{topo}}^{\text{Kitaev}} \stackrel{L\to\infty}{\longrightarrow} -\ln 2 \tag{61}$$

for any non-zero value of T, that is *thermal equilibrium at any infinitesimal temperature leads to a finite loss of topological entropy in the thermodynamic limit*. In Sec. VI we discuss the implications of this result, and in particular, we propose an interpretation that naturally explains why the topological entropy reduces to precisely half of its zero-temperature value $S_{\text{topo}}^{\text{Kitaev}} = 2\ln 2$.

Notice that Eq. (54) is consistent with both the zero-temperature and the infinite-temperature limits, Eqs. (52,53). Notice also that the topological entropy in the $L \to \infty$ limit becomes a pure function of $K_A \Sigma_{1\mathcal{P}_1}/2$, whose shape is illustrated in Fig. 4 The location of the drop, say when $S_{\text{topo}}(T) - S_{\text{topo}}^{\text{Kitaev}} = -(\ln 2)/2$, is given by

$$\frac{K_A}{2}\,\Sigma_{1\mathcal{P}_1} \simeq \frac{1}{4}. \tag{62}$$

Even for modest partition sizes with $\Sigma_{1\mathcal{P}_1} \gtrsim 100$, the drop occurs at rather small temperatures and we can approximate

$$K_A = -\ln\left[\tanh\left(\frac{\lambda_A}{T}\right)\right] \simeq 2e^{-2\frac{\lambda_A}{T}}. \tag{63}$$

This in turn gives

$$\Sigma_{1\mathcal{P}_1} e^{-2\frac{\lambda_A}{T_{\text{drop}}}} \simeq \frac{1}{4} \quad \Longrightarrow \quad T_{\text{drop}} \simeq \frac{\lambda_A}{\ln\left(2\sqrt{\Sigma_{1\mathcal{P}_1}}\right)}. \tag{64}$$

The l.h.s. of the above equation allows for a straightforward interpretation in terms of defects in the underlying electric loop structure. In fact, $e^{-\lambda_A/T}$ controls the density of such defects in the system, and the equation therefore suggests that the drop in topological entropy occurs when the average number of defects inside partition $1\mathcal{B}_1$ becomes of order one.

In order to understand the behavior of $S_{\text{topo}}(T)$ at finite temperature and finite system size, notice that the temperature parameter $K_A = -\ln[\tanh(\beta\lambda_A)]$ in Eq. (51) always appears multiplied by an extensive quantity, be it N or one of the Σ's. It is therefore convenient to make the reasonable assumption that the number of star operators in each subsystem $\mathcal{A}_1, \ldots, \mathcal{A}_{m_\mathcal{A}}$, and $\mathcal{B}_1, \ldots, \mathcal{B}_{m_\mathcal{B}}$ scales linearly with the total number of star operators N. Namely, this amounts to increasing uniformly both L and r, R while keeping their ratios fixed, thus simply rescaling the bipartitions in Fig. 3. We can then introduce the notation $\Sigma_\mathcal{A} = N\gamma_\mathcal{A}$, and $\Sigma_{\mathcal{P}_i} = N\gamma_{\mathcal{P}_i}$, with $\gamma_\mathcal{A}, \gamma_{\mathcal{P}_i} \in (0,1)$, and $\gamma_\mathcal{A} + \sum_i \gamma_{\mathcal{P}_i} = 1$. Recalling the definitions of $\tilde{x}_i^{(\alpha)} = \cosh(K_A\Sigma_{\alpha\mathcal{P}_i}/2)$ and $\tilde{y}_i^{(\alpha)} = \sinh(K_A\Sigma_{\alpha\mathcal{P}_i}/2)$, one can replace K_A by $\mathcal{K}_\mathcal{A} = K_A N$ and all other parameters in Eq. (51) become intensive quantities that do not scale with the system size. Temperature and system size are strongly bound together into a single tunable parameter $\mathcal{K}_\mathcal{A}$ in our system. The thermodynamic limit at zero temperature is singular, in that the behavior of $\mathcal{K}_\mathcal{A}$ depends on the order of limits.

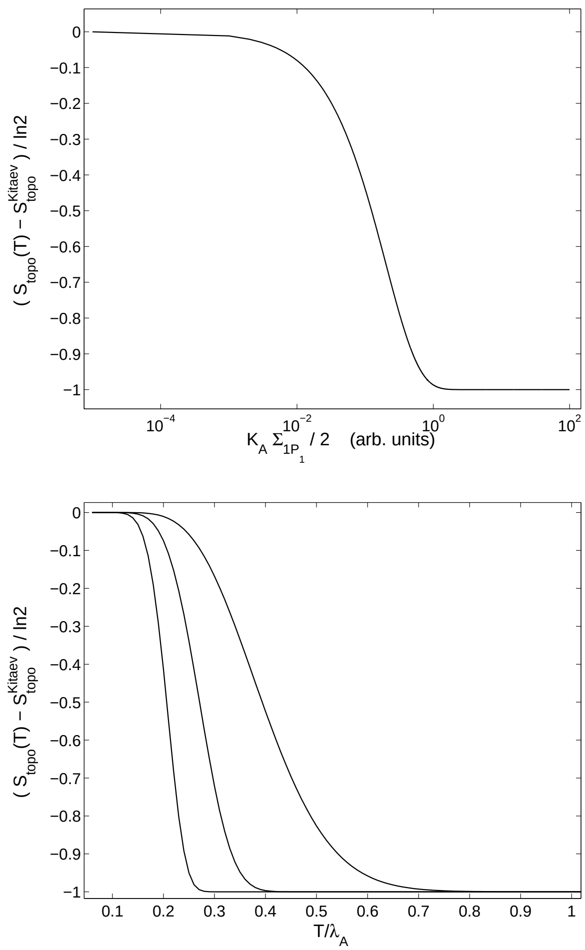

FIG. 4: (Top) Limiting behavior of the entropy difference Eq. (54) in units of $\ln 2$ in the thermodynamic limit, as a function of $K_A\Sigma_{1\mathcal{P}_1}/2$, where $K_A = -\ln[\tanh(\lambda_A/T)]$ and $\Sigma_{1\mathcal{P}_1} \sim (R-2r)^2$, the area of the inner square in Fig. 3. Notice the logarithmic scale on the horizontal axis. (Bottom) The same curve represented as a function of T/λ_A, for three different values of $\Sigma_{1\mathcal{P}_1} = 20, 200, 2000$ (from right to left).

A. Numerical evaluation of $S_{\text{topo}}(T)$

The expression for the topological entropy as a function of temperature and system size Eq. (51) is rather lengthy and non-transparent. In this section we illustrate its behavior graphically, by explicitly evaluating $S_{\text{topo}}(T)$ for small systems. In Fig. 5 we present the difference $(S_{\text{topo}}(T) -$

$S_{\rm topo}^{\rm Kitaev})/\ln 2$ as a function of $\mathcal{K}_{\mathcal{A}} = K_A N$, for various system sizes $N = 10^3, 10^6, 10^9$. For convenience, we chose the

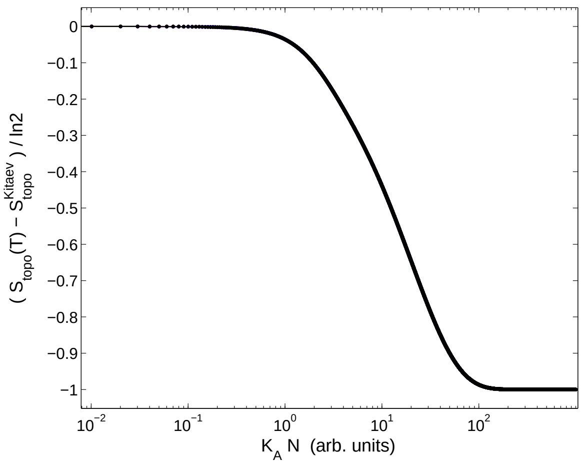

FIG. 5: Topological entropy as a function of $\mathcal{K}_{\mathcal{A}}$ for increasing system sizes $N = 10^3, 10^6, 10^9$. Notice the complete overlap between the different curves, due to the fact that the topological entropy Eq. (51) becomes a pure function of $\mathcal{K}_{\mathcal{A}} = K_A N$ when all Σ's scale linearly with N. Notice the logarithmic scale on the horizontal axis.

values of r and R proportional to $\sqrt{N}$, so that the above assumption on the Σ's holds true, and $S_{\rm topo}(T)$ is a function of $\mathcal{K}_{\mathcal{A}}$ only.

In the limit $L \to \infty$, the smooth curves collapse identically onto their infinite-temperature value for any non-vanishing temperature, and a singularity arises at $T = 0$.

The location of the drop is given by $K_A N \simeq 10$, from which we obtain

$$-\ln\left[\tanh\left(\frac{\lambda_A}{T_{\rm drop}}\right)\right] N \;\simeq\; 10$$
$$T_{\rm drop} \;\simeq\; \frac{\lambda_A}{\tanh^{-1}\left(e^{-\frac{10}{N}}\right)}. \qquad (65)$$

For large enough system sizes, $T_{\rm drop}$ is small and the above equations reduce to

$$N e^{-2\frac{\lambda_A}{T_{\rm drop}}} \simeq 5 \quad \Longrightarrow \quad T_{\rm drop} \simeq \frac{\lambda_A}{\ln\sqrt{N/5}}. \qquad (66)$$

Once again, the drop occurs when the average number of defects in the system becomes of order one. (This is consistent with the previous result in Eq. (64) since we made here the assumption that all the Σ's, and therefore $\Sigma_{1\mathcal{P}_1}$ as well, scale linearly with N).

V. THE FULL TEMPERATURE RANGE

In the regime considered in this paper, finite temperature disrupts the σ^x-loop structure gradually for finite size systems until it is completely destroyed. This happens while the σ^z-loop structure is fully preserved, and the topological entropy changes overall from $2\ln 2$ to $\ln 2$ (half of the contribution is lost).

The remaining topological entropy should fade away as temperature is further increased, and one goes to the regime where defects in the σ^z-loop structure also start to appear, for a finite energy scale λ_B. The temperature scale of the drop in $S_{\rm topo}$ from $\ln 2$ to 0 corresponds to when the distance between defects, $\xi_B \sim e^{\lambda_B/T}$, becomes comparable to the system size L. (Or equivalently, the average number of defects in the system becomes roughly of order one – compare with Eq. (64) and (66).)

It is not obvious how to obtain the exact expression for this second step, in contrast with the first step which we calculated exactly in this paper within the preserved σ^z-loop limit. Nevertheless, we believe that the physical picture is the simple one (as seen at work in the first drop) that once a handful of defects appear in that σ^z-loop structure, the topological entropy will plunge much like in the first drop. Pasting the two pictures together, we have the two-stage drop of the topological entropy sketched in Fig. 1. Clearly, in the limit $|\lambda_A - \lambda_B| \to 0$ the two drops are expected to merge together, and in particular in the thermodynamic limit the topological entropy entirely vanishes for any infinitesimal temperature.

We would like to point out that a notion of fragility in the Kitaev model at finite temperature, in terms of expectation values of toric operators, has been discussed by Nussinov and Ortiz[16] within their definition of topological quantum order (based on gauge-like symmetries).

VI. CONCLUSIONS

We calculated the entanglement entropy exactly for the toric code at finite temperatures, in a regime where there is a broad separation of energy scales between the two couplings in the problem, $\lambda_A \ll \lambda_B$. These couplings, from a $\mathbb{Z}_2$ gauge theory perspective, correspond to the chemical potentials of electric charges and magnetic monopoles. One can define length scales associated with the separation between these types of defects, $\xi_{A,B} \sim e^{\lambda_{A,B}/T}$, and for system sizes much smaller than the largest of these two length scales, i.e., $L \ll \xi_B$, one of the two loop structures in the system, associated with the σ^z-basis, is preserved. This is the regime where magnetic monopoles are not present in the finite size system. In the limit $\lambda_B \to \infty$, this holds true for any system size. It is in this limit that we obtain the exact result for the entanglement entropy as a function of T/λ_A.

Within this hard constrained regime, we find that the entanglement entropy is a singular function of temperature and system size, and that the limit of zero temperature and the limit of infinite system size do not commute. The two limits differ by a term that does not depend on the size of the boundary between the partitions of the system into two entangled parts, but instead depends on the topology of the bipartition. We also calculate the mutual information, obtained from the von Neumann entropy by a symmetrization procedure to filter

bulk terms at non-zero temperatures and to leave only boundary and topological contributions. Similarly, the difference between the two orders of limits is an $\mathcal{O}(1)$ term that is purely topological, depending on the number of disconnected pieces of partitions $\mathcal{A}$ and $\mathcal{B}$.

We find that one half of the topological entropy is shaved off from its $T = 0$ value as the temperature increases above $T^{(A)}_{\rm cross} \sim \lambda_A / \ln\sqrt{N}$. Above this scale, the loop structure associated with the $\sigma^{\rm x}$-basis is destroyed, while the one associated with the $\sigma^{\rm z}$-basis survives (recall the $\lambda_B \to \infty$). We argue that a large but finite value of λ_B would introduce another scale $T^{(B)}_{\rm cross} \sim \lambda_B / \ln\sqrt{N}$, above which the rest of the topological entropy should also vanish.

As these results show, the topological contributions to the von Neumann entropy or equivalently to the topological entropy, are rather fragile for non-zero temperatures. If the thermodynamic limit is taken first, these quantities subside immediately. However, in practice one should focus on physical regimes and not mathematical limits. The reason why these quantities are so fragile is that $\mathcal{O}(1)$ defects can destroy them. However, one must realize that the length scale associated to the defect separation grows exponentially as temperature is decreased, and becomes astronomical for temperatures a few hundred times smaller than the energy scales $\lambda_{A,B}$. Hence, even if these topological contributions to the entanglement entropy technically vanish, they are statistically present in large but laboratory size physical systems.

If one is interested in understanding how robust is the topological order (information) stored in a single finite system, the notion of a statistically non-vanishing topological entropy naturally translates into the presence of a characteristic time scale over which topoogical order is preserved. Such time scale is associated with the Boltzman probability for the appearance of a defect, namely $\mathcal{N}\, e^{-\lambda_A/T}$, where $\mathcal{N}$ is the total number of degrees of freedom in the system. In sight of a possible practical use of such topological quantum information, it would therefore be of great importance to compare this persistence time scale with the one associated to the preparation of the system into a topologically ordered state. Preliminary research in that direction can be found in Ref. 10 and in Ref. 17.

At a more fundamental level, our results suggest a simple pictorial interpretation of quantum topological order, at least for systems where there is an easy identification of loop structures as in the case here studied. Recall that we start from a zero-temperature system exhibiting quantum topological order associated with the presence of two identical underlying closed-loop structures. In particular, the corresponding topological entropy equals $\ln D^2$, where $D = 2$ is the so-called quantum dimension of the system. By allowing one of the two loop structures to be thermally disrupted, and by raising $T \to \infty$ while the other loop structure is fully preserved, we arrive at a classical system with a single (therefore classical) underlying loop structure, and exhibiting precisely half of the original topological entropy ($\ln D$). This is strongly suggestive that (i) the two loops structures contribute equally and independently to the topological order at zero temperature; (ii) each loop structure *per se* is a classical (non-local) object carrying a contribution of $\ln D$ to the topological entropy; and (iii) the quantum nature of the zero-temperature system resides in the fact that two independent loop structures are allowed to be superimposed and thus coexist in the system. In this sense, our results lead to an interpretation of quantum topological order, at least for systems with simple loop or membrane structures, as the quantum mechanical version of a classical topological order (given by each individual loop structure).

Finally, we would like to comment on the fact that the same $\mathcal{O}(1)$ defects that deteriorate the topological entropy of the system should also deteriorate its usefulness for topological quantum computing. A handful of stray unaccounted defects winding and braiding around others that are accounted for in the computational scheme will lead to errors. These defects can be thermally suppressed, if the temperature is small enough and the system not too large, so that unwanted defects have a small probability of appearing in the sample. Thus, quantifying topological entropy at finite temperature and finite system size is meaningful in quantifying, in a statistical sense, the degree with which a physical (finite) system retains topological order.

Although the results presented here were derived in the case of one of the coupling constants being infinite, we have recently been able to extend the calculations to the case where both coupling constants are finite [C. Castelnovo and C. Chamon, in preparation]. The two contributions to the topological entropy due to the underlying gauge structures are shown to behave additively, and indeed the behavior conjectured in Fig. 1 is confirmed.

Acknowledgments

We are grateful to Xiao-Gang Wen for his insightful comments on the loop structure underlying our model, and to Eduardo Fradkin for enlightening discussions. This work is supported in part by the NSF Grant DMR-0305482 (C. Chamon), and by EPSRC Grant No. GR/R83712/01 (C. Castelnovo). C. Castelnovo would like to acknowledge the I2CAM NSF Grant DMR No. 0645461 for travel support, during which part of this work was carried out.

1 F. D. M. Haldane, and E. H. Rezayi, Phys. Rev. B **31**, 2529 (1985).
2 X.-G. Wen, and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
3 X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990); Adv. in Phys. **44**, 405 (1995); Phys. Rev. B **65**, 165113 (2002).
4 D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
5 M. Levin, and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
6 A. Y. Kitaev, and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).

[7] M. Haque, O. Zozulya and K. Schoutens, Phys. Rev. Lett. **98**, 060401 (2007).

[8] C. Castelnovo and C. Chamon, arXiv:cond-mat/0610316 (2006) – accepted for publication in Phys. Rev. B.

[9] A. Y. Kitaev, Ann. Phys. (N.Y.) **303**, 2 (2003).

[10] A. Hamma and D. A. Lidar, arXiv:quant-ph/0607145v4 (2006).

[11] This limit can be obtained, for example, if the thermal bath is not allowed to couple to the individual degrees of freedom but to (local) products of them (namely, the star operators introduced in Sec. II). Such thermal bath has been previously discussed by Trebst *et al.* [Phys. Rev. Lett. **98**, 070602 (2007)] in the thermodynamic limit, and the authors concluded that no finite-temperature quantum phase transition is to be expected, and that one of the two loop strucutures is indeed preserved for any value of the dissipation strength.

[12] We are indebted to Xiao-Gang Wen for pointing out to us the possibility of a much richer behavior in higher dimensions, due to the different nature of the loop and membrane structures underlying topological order.

[13] A. Hamma, R. Ionicioiu, and P. Zanardi, Phys. Rev. A **71**, 022315 (2005).

[14] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974); R. Balian, J. M. Drouffe, and C. Itzykson, Phys. Rev. D **11**, 2098 (1975); E. Fradkin and L. Susskind, Phys. Rev. D **17**, 2637 (1978); E. Fradkin and S. Raby, Phys. Rev. D **20**, 2566 (1979); L. Susskind, Phys. Rev. D **20**, 2610 (1979).

[15] Notice that, while the constraint $\Sigma_{\mathcal{A}} + \sum_i \Sigma_{\mathcal{P}_i} = N$ requires at least one of the Σ's to diverge for $L \to \infty$, this does not need to be the case for all of them, in general.

[16] Z. Nussinov, and G. Ortiz, arXiv:cond-mat/0605316v2 (2006), and arXiv:cond-mat/0702377 (2007).

[17] R. Alicki, M. Fannes, and M. Horodecki, J. Phys. A: Math. Theor. **40**, 6451 (2007).